HW 3

PHYS 425/525

1. Griffiths Problem 3.2

As a STABLE mechanical potential function in all three directions has $\left(k_x x^2 + k_y y^2 + k_z z^2\right)/2$ with all three $\left(k_x, k_y, k_z\right)$ positive, because Laplace's equation for a real electrostatic potential around an equilibrium has $k_x + k_y + k_z = 0$, an electrostatic potential cannot be stable in all three directions.

Around, the center of the cube in Fig. 3.4, the Taylor series for the potential may be found by systematically taking second derivatives. For example

$$\frac{\partial^2}{\partial x^2}\frac{1}{\sqrt{\left(x \pm d/2\right)^2 + \left(y \pm d/2\right)^2 + \left(z \pm d/2\right)^2}}\Bigg|_{(0,0,0)}$$

$$= \frac{3\left(x \pm d/2\right)^2 - \left(x \pm d/2\right)^2 - \left(y \pm d/2\right)^2 - \left(z \pm d/2\right)^2}{\left(\left(x \pm d/2\right)^2 + \left(y \pm d/2\right)^2 + \left(z \pm d/2\right)^2\right)^{5/2}}\Bigg|_{(0,0.0)}$$

$$= 0$$

$$\frac{\partial^2}{\partial x \partial y}\frac{1}{\left(\left(x \pm d/2\right)^2 + \left(y \pm d/2\right)^2 + \left(z \pm d/2\right)^2\right)^{5/2}}\Bigg|_{(0,0,0)}$$

$$= 3\frac{\left(x \pm d/2\right)\left(y \pm d/2\right)}{\sqrt{\left(x \pm d/2\right)^2 + \left(y \pm d/2\right)^2 + \left(z \pm d/2\right)^2}}\Bigg|_{(0,0.0)}$$

$$= \frac{3}{3^{5/2}}\frac{(\pm)(\pm)2^3}{d^3}$$

So, summing over all the charges leads to a vanishing of the second derivative terms in the Taylor expansion of the potential and so $k_x = k_y = k_z = 0$. This potential is not guaranteed to be stable.

2. Griffiths Problem 3.9

The main point of this problem is to realize that the SAME image charge pair applies here, as applies in Example 3.2, but now the potential INSIDE the spherical surface is relevant. For this problem let

$$b = R^2 / a > R$$

$$q' = -\frac{R}{a} q$$

$$\phi(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{Rq}{a\sqrt{x^2+y^2+\left(z-R^2/a\right)^2}}\right]$$

When $x^2+y^2+z^2=R^2$, a bit of algebra shows $\phi(x,y,z)=0$. By the uniqueness theorem this is the potential inside a grounded spherical shell. By symmetry, the force is entirely in the *z*-direction. By differentiation or by just applying Coulomb's Law it is

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{R}{a}\frac{q^2}{\left(a-R^2/a\right)^2}\hat{z} = \frac{1}{4\pi\varepsilon_0}\frac{Raq^2}{\left(a^2-R^2\right)^2}\hat{z}$$

The force attracts the charge to the surface, and vanishes when $a \to 0$. For a general displacement not in the *z*-direction, the force is just radially out! Also note, the answer is the negative of the answer in Example 3.2 (Equation 3.18), as it must!

3. Griffiths Problem 3.14

First note the left-right symmetry about the *y*-axis is the same as in Problem 2.53. So we should place two "image" charges with opposite signs at the same distance from the *y*-axis. Now if $\pm a$ are the positions of the line charges, we need to choose parameters so that the center of the equipotential circle is *d* and its radius is *R*. Thus, from Problem 2.53

$$R = \frac{a}{\sinh\left(2\pi\varepsilon_0\phi_0/\lambda\right)} \qquad d = a\frac{\cosh\left(2\pi\varepsilon_0\phi_0/\lambda\right)}{\sinh\left(2\pi\varepsilon_0\phi_0/\lambda\right)} \to \cosh^{-1}\frac{d}{R} = \frac{2\pi\varepsilon_0\phi_0}{\lambda}$$

The total potential is

$$\phi = \frac{\phi_0}{\cosh^{-1}(d/R)}\left[-\ln\sqrt{(x-a)^2+y^2} + \ln\sqrt{(x+a)^2+y^2}\right]$$

$$= \frac{\phi_0}{2\cosh^{-1}(d/R)}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2} = \frac{\phi_0}{2\cosh^{-1}(d/R)}\ln\frac{\left(x+\tanh\left(2\pi\varepsilon_0\phi_0/\lambda\right)d\right)^2+y^2}{\left(x-\tanh\left(2\pi\varepsilon_0\phi_0/\lambda\right)d\right)^2+y^2}$$

But if $\cosh x = y$, then

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\sqrt{\cosh^2 x - 1}}{\cosh x}.$$

So

$$\tanh\left(2\pi\varepsilon_0\phi_0 / \lambda\right)d = \frac{\sqrt{\left(d/R\right)^2 - 1}}{d/R}d = \sqrt{d^2 - R^2}$$

and

$$\phi = \frac{\phi_0}{2\cosh^{-1}\left(d/R\right)}\ln\frac{\left(x+\sqrt{d^2-R^2}\right)^2 + y^2}{\left(x-\sqrt{d^2-R^2}\right)^2 + y^2}.$$

A straightforward calculation using $\cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right)$ shows $\phi\left(circles\right) = \pm\phi_0$. As this solution matches the boundary conditions, by the uniqueness theorem it is the correct solution.

4. Griffiths Problem 3.19

$$\begin{aligned}
P_3\left(x\right) &= \frac{1}{8\cdot 6}\left(\frac{d}{dx}\right)^3\left(x^2-1\right)^3 \\
&= \frac{1}{8}\left(\frac{d}{dx}\right)^2 x\left(x^2-1\right)^2 \\
&= \frac{1}{8}\frac{d}{dx}\left[\left(x^2-1\right)^2 + 4x^2\left(x^2-1\right)\right] \\
&\frac{1}{2}x\left(x^2-1\right) + x\left(x^2-1\right) + x^3 = \frac{5x^3-3x}{2}
\end{aligned}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}P_3\left(\cos\theta\right)\right) = \frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\frac{5\cos^3\theta - 3\cos\theta}{2}\right)$$

$$= \frac{d}{d\theta}\left(\sin^2\theta\frac{-15\cos^2\theta+3}{2}\right) = 2\sin\theta\cos\theta\frac{-15\cos^2\theta+3}{2} + 2\sin^3\theta\frac{15\cos\theta}{2}$$

$$= \sin\theta\left(-15\cos^3\theta + 3\cos\theta + 15\cos\theta - 15\cos^3\theta\right) = \sin\theta\left(3\cdot 4\right)\frac{-30\cos^3\theta + 18\cos\theta}{\left(3\cdot 4\right)}$$

$$= -\sin\theta\left(3\cdot 4\right)P_3\left(\cos\theta\right)$$

$$\int_{-1}^{1} P_1\left(x\right)P_3\left(x\right)dx = \int_{-1}^{1} x\frac{5x^3-3x}{2}dx$$

$$= \left.\frac{x^5}{2}\right|_{-1}^{1} - \left.\frac{x^3}{2}\right|_{-1}^{1} = 1 - 1 = 0$$

5. Griffiths Problem 3.25

Here we simply apply Equation 3.84 from Example 3.9 for the expansion coefficients.

$$A_l = \frac{1}{2\varepsilon_0 R^{l-1}} \int_{-1}^{1} \sigma(\theta) P_l(\cos\theta)\, d\cos\theta$$
$$= \frac{\sigma_0}{2\varepsilon_0 R^{l-1}} \left[\int_{0}^{1} P_l(\cos\theta) d\cos\theta - \int_{-1}^{0} P_l(\cos\theta) d\cos\theta \right]$$

For even *l*, note that the integrands are unchanged under the replacement $x \to -x$, and so the integrals are identical and cancel. For odd *l*, the integrands and integrals are negatives under the replacement, so for odd *l*

$$A_l = \frac{\sigma_0}{\varepsilon_0 R^{l-1}} \int_0^1 P_l(x) dx$$

$$A_1 = \frac{\sigma_0}{\varepsilon_0} \left. \frac{x^2}{2} \right|_0^1 = \frac{\sigma_0}{2\varepsilon_0}$$

$$A_3 = \frac{\sigma_0}{\varepsilon_0 R^2} \left[\left. \frac{5x^4}{8} \right|_0^1 - \left. \frac{3x^2}{4} \right|_0^1 \right] = -\frac{\sigma_0}{8\varepsilon_0 R^2}$$

$$A_5 = \frac{\sigma_0}{\varepsilon_0 R^4} \left[\left. \frac{63x^6}{48} \right|_0^1 - \left. \frac{70x^4}{32} \right|_0^1 + \left. \frac{15x^2}{16} \right|_0^1 \right] = \frac{\sigma_0}{\varepsilon_0 R^4} \frac{126 - 210 + 90}{96} = \frac{\sigma_0}{16\varepsilon_0 R^4}$$

By equation 3.81

$$B_1 = A_1 R^3 = \frac{\sigma_0 R^3}{2\varepsilon_0} \quad B_3 = A_3 R^7 = -\frac{\sigma_0 R^5}{8\varepsilon_0} \quad B_5 = A_5 R^{11} = \frac{\sigma_0 R^7}{16\varepsilon_0}$$

and the even B_l vanish. Note the proper units for the potential obtain, namely Nt m/C.

6. Griffiths Problem 3.18 (Extra Credit) This problem has general 3-D solution to Laplace's Equation of

□ □ □ □ □ □ 2 2 , 1 1 sin / sin / sinh / m n m n A mx a ny a m n z a □ □ □ □ □ □ □ □ □□

which automatically solves the boundary conditions given in Figure 3.23. Why? The rest of the solution follows the methods of Example 3.5.

$$\exp\left(-s^2 / 2\sigma_s^2\right)$$